1.5

1

Important differential operators:

1) Divergence.

If $\vec{F}(x,y,z) = \left(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \right)$

$\vec{x} = (x,y,z)$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x}(\vec{x}) + \frac{\partial F_2}{\partial y}(\vec{x}) + \frac{\partial F_3}{\partial z}(\vec{x}).$$

$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longrightarrow \vec{F}(x,y,z).$

2) Gradient

If $f(x,y,z)$ is a Scalar function of 3 variables

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longrightarrow f(x,y,z)$$

$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x}(\vec{x}), \frac{\partial f}{\partial y}(\vec{x}), \frac{\partial f}{\partial z}(\vec{x}) \right)$$

3) Laplacian.

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Proof.-

$$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) =$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \checkmark$$

Divergence Theorem.

Concept of Region:- An open set containing all or some of the points forming its boundary.

Def.- A closed surface $\partial\Omega$ that consists of a finite number of smooth pieces joined together at the boundaries (curves defining boundaries) is called closed piecewise-smooth surface.

Def.- By a smooth surface S, we mean

$$S: \vec{r} = \vec{r}(u,v), \quad (u,v) \in D.$$

Such that $\vec{r}(u,v)$ is continuously differentiable and its unit normal vector $\hat{n}(\vec{x}_s)$ is continuous on S.

Theorem.- a) Ω is a bounded region.

b) $\partial\Omega$ is the closed piecewise smooth surface of Ω.

c) $F(\vec{x})$ is continuous on $\Omega \cup \partial\Omega$.

d) $F(\vec{x})$ is continuously differentiable in Ω.

e) $\hat{n}(\vec{x}_s)$ is the unit outer normal vector to Ω at $\vec{x}_s$.

Then,

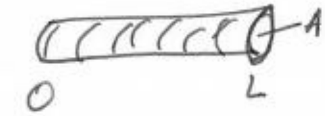
$$\iiint_{\Omega} \nabla \cdot \vec{F}(\vec{x})\, dv = \iint_{\partial\Omega} (\vec{F} \cdot \hat{n})(\vec{x}_s)\, ds.$$

Fundamental theorem of Calculus as a particular case of Divergence theorem.

Consider the divergence theorem

omit if not much time

$$\iiint_{\Omega} \nabla\cdot\vec{F}\, dv = \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, ds$$



1) If $\Omega \equiv$ Finite cylinder of constant cross-section A and lenght L.

2) If $\vec{F} = (F_1, F_2, F_3) = (F_1, 0, 0)$

and F_1 is constant at any cross section

then $F_1(x,y,z) = \bar{F}_1(x)$

$$\text{and} \iiint_{\Omega} \nabla\cdot F\, dv = \iiint_{\Omega} \frac{\partial F_1}{\partial x}(x,y,z)\, dv = A\int_0^L \frac{d\bar{F}_1}{dx}\, dx =$$

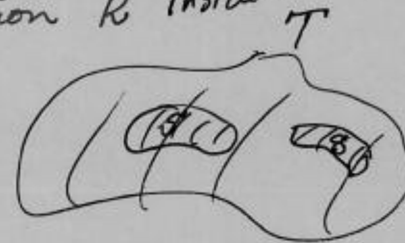
$$= A\left[\bar{F}_1(L) - \bar{F}_1(0)\right] \quad \checkmark$$

$$\text{Also,} \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, ds = \iint_{\text{walls}} \vec{F}\cdot\hat{n}\, ds + \iint_{\text{at } x=L} \vec{F}\cdot\hat{n}\, ds + \iint_{\text{at } x=0} \vec{F}\cdot\hat{n}\, ds$$

At walls: $\vec{F}\cdot\hat{n} = (F_1, 0, 0)\cdot\hat{n} = 0$, At $x=L$ $\vec{F}\cdot\hat{n} = \bar{F}_1(L)$, At $x=0$ $\vec{F}\cdot\hat{n} = -\bar{F}_1(0)$

$$\Rightarrow \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, ds = \bar{F}_1(L)\iint_{x=L} ds - \bar{F}_1(0)\iint_{x=0} ds = A\left[\bar{F}_1(L) - \bar{F}_1(0)\right] \checkmark$$

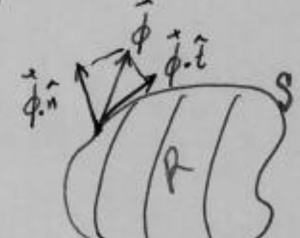
In 2-D or 3-D, Consider an arbitrary region R inside T



Want to study conservation of heat energy inside Region R bounded by surface S.

(I) Heat energy inside R $= \iiint_R c(x,y,z)\,\rho(x,y,z)\,v(x,y,z,t)\,dv$

(II) ϕ or heat flux is now a vector function $\vec{\phi}(x,y,z,t)$.



If $\vec{\phi}.\hat{n}>0 \Rightarrow \phi$ is directed outward of R

If $\vec{\phi}.\hat{n}<0 \Rightarrow \phi$ is directed inward of R

$\hat{n}$: outward unit normal.

Flux of heat energy out of region R $= \iint_S \vec{\phi}(x,y,z,t).\hat{n}(x,y,z)\,ds$

$$\overset{\text{Div thm}}{=} \iiint_R \nabla.\vec{\phi}(x,y,z,t)\,dv$$

Heat energy per unit of Vol. generated inside R and per unit of time. $= Q(x,y,z,t)$.

Therefore, Conservation of energy inside R leads to

$$\frac{\partial}{\partial t}\iiint_R c\rho v\,dv \overset{\text{Ask?}}{=} -\iint_S \vec{\phi}.\hat{n}\,ds + \iiint_R Q(\vec{x},t)\,dv$$

$$\overset{\text{Div Thm}}{=} -\iiint_R \nabla.\vec{\phi}\,dv + \iiint_R Q\,dv$$

or

$$\iiint_R \left[c\rho \frac{\partial v}{\partial t} + \nabla \cdot \phi - Q \right] dv = 0$$

True for any region R in Γ, then

$$c\rho \frac{\partial v}{\partial t} = -\nabla \cdot \phi + Q$$

Again, Fourier's law of Heat conduction:

$\vec{\phi} = -K_0 \nabla v$ Same physical properties as in 1-D.

$$\Rightarrow \quad \boxed{c\rho \frac{\partial v}{\partial t} = -\nabla \cdot (-K_0 \nabla v) + Q.} \qquad (2.1)$$

If thermal conductivity K_0 is constant and $Q = 0$. (no sources of Heat energy)

$$\frac{\partial v}{\partial t} = k \ \nabla \cdot (\nabla v)$$

then $\boxed{\frac{\partial v}{\partial t} = k \nabla^2 v}$ Heat or diffusion equ. (2.2)

Possible BC's.: $v(x,y,z,t) = T(x,y,z,t)$ on boundary S.

$$\frac{\partial v}{\partial n} = \nabla v \cdot \hat{n}\,(x,y,z,t) = H(x,y,z,t) = \begin{cases} \neq 0 \\ = 0, \text{ insulated.} \end{cases}$$

IC : $v(x,y,z,0) = f(x,y,z)$. → Equations of elliptic type.

Steady state + No heat sources + Constant thermal properties in (2.1) leads to

$$\boxed{\begin{array}{ll} \nabla^2 v = \dfrac{-Q}{K_0} & \text{Poisson's equ.} \\ \nabla^2 v = 0, & \text{Laplace's equation} \end{array}}$$

If $Q = 0$

Discuss analytic techniques and limitations. !

Fourier's Law of heat Conduction in Higher dimensions

In 1-d

$$\text{Flux} = \phi(x,t) = -K_0(x)\,\frac{\partial u}{\partial x}(x,t).$$

In 2-d or 3-d.

The flux is a vector $\vec{\phi}(\vec{x},t) = \vec{\phi}(x,y,z,t)$.

The question is in what direction the heat energy is naturally flowing.

<u>answer</u>: For isotropic materials, "heat flows from hotter to colder regions in the direction in which temperature differences are greatest."

In math terms, heat flows in the direction in which the highest value of the rate of change of u among all directional derivatives is achieved.

So

$$\boxed{\vec{\phi}(\vec{x},t) = -K_0(\vec{x})\,\nabla u}$$

In next two pages, the direction of maximum rate of change is determine.

Rate of change of temperature $u(\vec{x},t)$ in a given direction $\hat{v}$ $(\|\vec{v}\|=1)$ is given by $u(\vec{x},t)$ directional derivative in the direction of $\vec{v}$.

It means

$$D_{\hat{v}}\, \hat{u}(x,t) = \nabla u(\vec{x},t)\cdot \vec{v}.$$

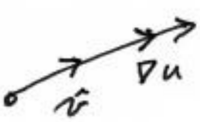
This rate of change is the greatest when $\vec{v} = \nabla u$.

In fact,

$$\nabla u \cdot \vec{v} = \|\nabla u\|\, \overbrace{\|\vec{v}\|}^{=1} \cos\theta$$

$$= \|\nabla u\| \cos\theta$$

This is greatest when $\theta = 0$



$$\Rightarrow \quad \hat{v} = \frac{\nabla u}{\|\nabla u\|}$$

Also, $\vec{\phi}$ is proportional to temp. differences

$$\Rightarrow \quad \vec{\phi}(\vec{x},t) = -K_0(\vec{x})\, \nabla u(\vec{x},t)$$

Example: — Consider $f(x,y) = x e^{y}$

a) Find the directional derivative of $f(x,y)$ at $(2,0)$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ $(1,0)$, $(0,1)$, $(1,2)$

b) In what direction does f have the maximum rate of change?. What is the maximum rate of change?

$$D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} f = \nabla f^{(2,0)} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(e^{y}, x e^{y}\right)\Big|_{(2,0)} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$D_{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)} f = \nabla f(2,0) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = (1,2) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}}$$
$$= \frac{5}{\sqrt{5}} = \sqrt{5}.$$

$$D_{(1,0)} f = \frac{\partial f}{\partial x}(2,0) = e^{0} = 1$$

$$D_{(0,1)} f(2,0) = \frac{\partial f}{\partial y}(2,0) = 2.$$

Largest value : $\boxed{D_{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)} f(2,0) = \sqrt{5}}$

b) In general, for $\vec{\omega}$ s.t. $\|\vec{\omega}\|=1$

$$D_{\hat{\omega}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{\omega}$$

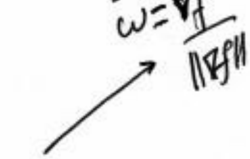
but

$$\nabla f(x_0,y_0) \cdot \hat{\omega} = \|\nabla f(x_0,y_0)\| \underbrace{\|\vec{\omega}\|}_{=1} \cos\theta$$

$$= \|\nabla f(x_0,y_0)\| \cos\theta$$

Largest value when $\theta = 0$.

$\Rightarrow$ $\vec{\omega}$ is in the direction of $\nabla f(x_0,y_0)$.



In our previous example,

$$\nabla f(2,0) = \left(e^y, xe^y\right)\Big|_{(2,0)} = (1,2).$$

So $(1,2)$ is the direction of maximum increase. for $f(x,y)$ at the point $(2,0)$

Maximum rate of change:

$$D_{\frac{\nabla f(2,0)}{\|\nabla f\|}} f(2,0) = D_{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)} f(2,0) = \nabla f(2,0) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= \sqrt{5}.$$